

# Study of Period Functions

Li Chengzhi

(School of Mathematical Sciences, Peking University, Beijing 100871, China)

**Abstract** In this survey article we first briefly introduce some concepts related to the period function of a planar smooth (or analytic) vector field, and its isochronicity, monotonicity, and the number of critical periods. Then, we introduce some important results in this field, especially about the isochronous centers, the period functions associated to the elliptic and hyperelliptic Hamiltonian functions, and the period functions of quadratic integrable systems. Besides these results we list some conjectures and problems in Section 6, which may provide topics for further studies.

**Key words** Period function Isochronicity Monotonicity Critical period

## 关于周期函数的研究

李承治

(北京大学数学科学学院, 北京, 100871)

**摘 要** 在这篇综述中我们首先简要介绍平面光滑 (或解析) 向量场的周期函数的概念, 以及它的等时性、单调性和临界周期的个数. 然后我们介绍本领域中一些重要的结果, 特别是关于等时中心, 关于椭圆函数或超椭圆函数所导出的向量场的周期函数, 和关于二次可积系统的周期函数. 除了这些结果, 我们在第六节中还列出一些相关的猜想和问题, 这可以作为进一步研究的论题.

**关键词** 周期函数 等时性 单调性 临界周期

**doi:** 10.3969/j.issn.1006-8074.2023.01.001

## 1 Introduction

### 1.1 The period function and related concepts

A continuous family of ovals  $\gamma_h \subset H^{-1}(h)$ , where  $H$  is the Hamiltonian function (or the first integral) of a smooth (or analytic) planar Hamiltonian (or integrable) system, forms an annulus (or annuli), which is (are) the largest neighbourhood(s) covered by the ovals. Each  $\gamma_h$  is a periodic orbit of the system with minimum period  $T$  depending on  $h$ , so we have a period function  $T(h)$ , parameterized by the same  $h \in (a, b)$ .

As  $h \rightarrow a_+$  (or  $b_-$ ), the orbit  $\gamma_h$  may shrink to a center of the vector field, may tend to a homoclinic orbit or heteroclinic orbits (or a union of them), also may tend to the whole equator or a part of it on the Poincaré disc. In the last case we have  $a = -\infty$  or  $b = +\infty$ .

If the period function is a constant for all  $h$ , i. e.,  $T'(h) \equiv 0$ , then the period annulus is called isochronous. If the isochronous period annulus surrounds a center, then the center is called isochronous. If the period function is strictly increasing or strictly decreasing, we say it is monotone. The isolated zeros of  $T'(h)$  correspond to critical periods.

Chicone and Dumortier [21] proved that if the vector field is analytic, its annulus is located in a compact region and is not isochronous, then the period function has a finite number of critical points (the zeros of  $T'(h)$ ), hence each critical point is isolated. Therefore, when the vector field is analytic, and  $T'(h) \geq 0$  (or  $T'(h) \leq 0$ ) and  $T'(h)$  is not equal to zero identically for  $h$  in a compact interval, then the period function is monotone increasing (or monotone decreasing). Moreover, from the multiplicity of  $h_0$  as a zero of  $T'(h)$  we can determine that the period function at  $h_0$  has a local minimum, maximum, or inflection point. In the last case, if the annulus depends on some other parameters, then the period function may have extreme points when the parameters change. The study of the behaviors of critical periods when the system depends on some parameter(s) is a difficult and complicated procedure, see the fundamental works by Chicone and Jacobs [22, 23], for example.

We will introduce more definitions and some general results concerning period functions in Section 2, about isochronous centers in Section 3, the behaviors of period functions related to elliptic and hyperelliptic Hamiltonians and quadratic integrable systems in Sections 4 and 5 respectively, and list some conjectures and problems in Section 6.

About 15 years before there was a short survey on period functions in Part II, Section 2.4 of [31], we want to contain more progresses in this field in the present article. We apologize for missing some interesting papers and for possible mistakes in this survey, and hope our readers to tell us about them for possible improvements in the future.

## 1.2 Why do we study the priod function?

In many applications the period function has a specific meaning, hence it is important to know that it is isochronous, or monotone, or it has local minimum or maximum point(s). The study of isochronous systems goes back at least to Galileo who discovered in 1632 the isochronicity of small oscillations of simple pendulum, and the formula for its period was given by Huygens in 1673, see page 72 of [114].

The study of the period function is also related to solutions of boundary value problems of differential equations, see Section 5 of [21] for example.

The study of the Abelian integral and the study of the period function have some relations, at least from the following two points of view.

Firstly, the study of the number of critical points of the period function by perturbing an isochronous center inside a certain class of integrable systems is comparable to the study of the number of zeros of an Abelian integral by perturbing an integrable system inside a certain class of systems.

Secondly, the study of the period function is useful for the study of Abelian integrals. For example, when the orientation of  $\gamma_h$  is counterclockwise, the Abelian integral  $I_0(h) = -\oint_{\gamma_h} y \, dx$  gives the area of the region surrounded by  $\gamma_h$ . Here  $\gamma_h \subset H^{-1}(h)$ , and the related Hamiltonian system is  $X_H =$

$-H_y \partial/\partial x + H_x \partial/\partial y$ , or (2.1) below. Hence,  $I_0(h) > 0$  for  $h > a$  and  $I'_0(h) > 0$  give the period of  $\gamma_h$ . In fact,

$$I'_0(h) = - \oint_{\gamma_h} \frac{\partial y}{\partial h} dx = - \oint_{\gamma_h} (H_y)^{-1} dx = \oint_{\gamma_h} dt = T(h).$$

If the period function is monotone, then  $I''_0(h) \neq 0$ . In some studies of Abelian integrals this information is needed to define a function by a ratio of two Abelian integrals with second order derivatives, see for example [9, 10, 26, 46, 47, 49, 72]. If  $I''_0(h)$  has zero or zeros, i.e., the period function has critical points, then the use of the ratio becomes complicated, see [48].

## 2 Preliminaries and some general results

(1) If  $H(x, y)$  is an analytic (or smooth) function, then the system

$$\frac{dx}{dt} = -\frac{\partial H(x, y)}{\partial y}, \quad \frac{dy}{dt} = \frac{\partial H(x, y)}{\partial x} \quad (2.1)$$

is called Hamiltonian. If  $H$  is a polynomial in  $x$  and  $y$ , then (2.1) is called a polynomial Hamiltonian system.

(2) If  $P(x, y)$  and  $Q(x, y)$  are analytic (or smooth), and the system

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \quad (2.2)$$

is not Hamiltonian, but there are analytic (or smooth) functions  $\tilde{H}(x, y)$  and  $\mu(x, y)$ ,  $\mu(x, y) > 0$  in the annulus we consider, such that

$$\mu P = -\frac{\partial \tilde{H}}{\partial y}, \quad \mu Q = \frac{\partial \tilde{H}}{\partial x},$$

then system (2.2) is called integrable, with integrating factor  $\mu$  and first integral  $\tilde{H}$ . In fact, in this case system (2.2) is equivalent (at least in the annulus) to a Hamiltonian system

$$\frac{dx}{d\tau} = -\frac{\partial \tilde{H}(x, y)}{\partial y}, \quad \frac{dy}{d\tau} = \frac{\partial \tilde{H}(x, y)}{\partial x},$$

with a time change. In particular, if  $P$  and  $Q$  are polynomials in  $x$  and  $y$ , then system (2.2) is called a polynomial system.

(3) We say that a critical point of system (2.2) is a center if there is a punctured neighborhood of the point which consists entirely of closed trajectories surrounding that point. The center is said to be non-degenerate if the linearized vector field at the point has two nonzero eigenvalues.

(4) We list some known results below.

- An isochronous center is non-degenerate, see Theorem 2.2 of [29].
- An isochronous period annulus cannot have any finite critical point on its outer boundary (of course, the isochronous center is its inner boundary), see Theorem 2.3 of [29].

- Consider the Hamiltonian system (2.1) with

$$H = \frac{x^2 + y^2}{2} + h_m(x, y), \quad (2.3)$$

where  $h_m(x, y)$  is a homogeneous polynomial of degree  $m \geq 3$ . System (2.3) has an isochronous center at the origin if and only if  $h_m(x, y) \equiv 0$ , see Theorem 5.6 of [29]. From Theorem B of [58] we know that in the statement “at the origin” can be deleted.

- A center is called *global*, if the annulus surrounding the center is the whole plane, except the center point. He and Xiao [68] characterized polynomial differential systems with a global center, and gave algebraic sufficient and necessary conditions for potential systems and Liénard systems which have a global center, respectively. He, Llibre and Xiao [69] gave the necessary and sufficient conditions for polynomial Hamiltonian systems to have a global center.

- (5) Gasull, Guillamon, Manósa and Manósas [58] investigated the behavior of the period function of system (2.3), where the homogeneous part  $h_m(x, y)$  is not identically zero. Let  $m = n + 1$ ,  $n \geq 2$ . Write system (2.3) in complex coordinates as

$$\dot{z} = iz + F_n(z, \bar{z}),$$

where  $z \in \mathbb{C}$ ,  $F_n(z, \bar{z}) = \sum_{k+l=n} f_{k,l} z^k \bar{z}^l$ ,  $\text{Re}[\partial F_n(z, \bar{z})/\partial z] = 0$ .

Let  $T(h)$  be the period function associated to the origin's period annulus of system (2.3). They obtained:

- (a)  $T(h)$  satisfies one of the following properties:

- (i) It is monotonic decreasing if and only if  $n$  is odd and

$$g(\theta) = h_{n+1}(\cos \theta, \sin \theta) \geq 0;$$

- (ii) It is monotonic increasing if and only if either  $n$  is even, or  $n$  is odd, and  $\text{Im}(f_{(\frac{n+1}{2}, \frac{n-1}{2})}) \leq 0$ ;

- (iii) It has a unique non-degenerate critical period if and only if  $n$  is odd,  $\text{Im}(f_{(\frac{n+1}{2}, \frac{n-1}{2})}) > 0$ , and there exists  $\theta \in [0, 2\pi)$  such that  $g(\theta) < 0$ .

- (b) There are systems of type (2.3) having a critical point of center type (different from the origin) for which the period function has at least two critical periods.

The results of [58] were extended by Álvarez, Gasull and Prohens [1] for Hamiltonians from  $x^2 + y^2$  plus homogenous to more general forms. Buzzi, Carvalho and Gasull [5] proposed a method to give the Taylor expansion of period function for Hamiltonian systems in terms of the energy (most known methods give this expansion in terms of the initial condition).

Garijo, Gasull and Jarque [54] studied properties of the period function for systems in complex notation as

$$\frac{dz}{dt} = f(z)g(\bar{z}), \quad z \in \mathbb{C}, t \in \mathbb{R},$$

and proved the existence of at most one critical period. The results were extended in their continued work [55].

- (6) In the particular form of (2.1), where  $H(x, y) = \frac{y^2}{2} + V(x)$  with  $V$  analytic (or smooth), the Hamiltonian system

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = V'(x), \quad (2.4)$$

is called a potential system. Especially, if  $V$  is a polynomial, then it is called a kinetic potential system.

In the case  $V(x) = \frac{x^2}{2} + o(x^2)$ , the center  $(0, 0)$  is non-degenerate, and the annulus  $\mathcal{P}$ , surrounding this center, can be given by ovals  $\{\gamma_h \subset H^{-1}(h) : h \in (0, h_0)\}$ , where  $h_0 = +\infty$  if the annulus is unbounded. If  $xV'(x) > 0$  for all  $x \in \mathcal{I} \setminus 0$ , where  $\mathcal{I}$  is the projection of  $\mathcal{P}$  on the  $x$ -axis, then each  $\gamma_h$  is convex, and we may define an analytic involution  $\sigma(x)$  by

$$V(\sigma(x)) = V(x) \quad \text{for all } x \in \mathcal{I}. \quad (2.5)$$

Recall that a mapping  $\sigma$  is said to be an involution if  $\sigma \circ \sigma = \text{Id}$  and  $\sigma \neq \text{Id}$ .

The earliest papers to use the idea of involution probably are [116] and [79], the former studied the isochronous problems and the latter studied the Abelian integrals. See also [65, 90].

- (7) Chicone [18] proved that if the potential system (2.4) has a non-degenerate local minimum at  $x = 0$ , and  $V/(V')^2$  is convex, then the period function is monotone. This condition was generalized in [4] and Section 5 of [35].
- (8) Schaaf [103] gave a criterion to determine the monotonicity of the period function for the potential system (2.4). We use the statement in [108] as follows.

The center at the origin of system (2.4) has a monotonous increasing period function in the case that

$$(a_1) \quad 5V'''(x)^2 - 3V''(x)V^{(4)}(x) > 0, \text{ for any } x \in \mathcal{I} \text{ with } V''(x) > 0,$$

and

$$(a_2) \quad V'(x)V'''(x) < 0, \text{ for any } x \in \mathcal{I} \text{ with } V''(x) = 0.$$

On the other hand, if

$$(b) \quad 5V'''(x)^2 - 3V''(x)V^{(4)}(x) < 0, \text{ for any } x \in \mathcal{I} \text{ with } V''(x) \geq 0,$$

then the period function is monotonous decreasing.

- (9) Gasull, Liu and Yang [60] proved that for each even  $n$  there is a polynomial system of degree  $n$  having a center such that the period function associated to its period annulus has at least  $\frac{1}{4}n^2 + \frac{3}{2}n - 4$  critical periods. There is a similar result when  $n$  is odd.

Later Cen [6] improved the above number to  $\frac{n^2}{2} - 2$  when  $n$  is even and  $\frac{n^2}{2} + n - \frac{5}{2}$  when  $n$  is odd. Recently, De Maesschalck and Torregrosa [43] improved the above number to  $n^2 - 2n$  when  $n$  is odd.

- (10) Mañosas and Villadelprat [90] gave a criterion to bound the number of critical periods of the potential system (2.4), where  $V(x) = \frac{1}{2m}x^{2m} + o(x^{2m})$  with  $m$  a positive natural number, with a period annulus  $\mathcal{P}$  surrounding the center at the origin.

They define

$$\mu_1 = -\frac{1}{2} + \left(\frac{V}{V'}\right)', \quad \mu_{k+1} = \frac{1}{2}\mu_k + \frac{1}{2k-1} \left(\frac{\mu_k V}{V'}\right)',$$

and

$$\mathcal{P}(\mu_k)(x) = \frac{\mu_k(x) - \mu_k(\sigma(x))\sigma'(x)}{2}, \quad x \in \mathcal{I} = (x_l, x_r),$$

where  $\sigma$  is defined in (2.5). Their criterion is as follows.

For any  $k \geq 1$ , one has

$$T'(h) = \frac{1}{h^k} \int_{\gamma_h} \mu_k(x) y^{2k-3} dx = \frac{1}{h^k} \int_{\gamma_h} \mathcal{P}(\mu_k)(x) y^{2k-3} dx.$$

Moreover, if for some  $k \geq 1$  the number of zeros of  $\mathcal{P}(\mu_k)(x)$  on  $(0, x_r)$ , counted with multiplicities, is  $n < k$ , then the number of critical periods (i.e., zeros of  $T'$ ), counted with multiplicities, is at most  $n$ .

The proof of this result strongly relies on a criterion given in [65] that provides a sufficient condition for a collection of Abelian integrals to be a Chebyshev system.

- (11) Sabatini [102] studied the period function for the Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \quad (2.6)$$

He changed (2.6) to an equivalent form

$$\dot{x} = y - xB(x), \quad \dot{y} = -C(x) - yB(x),$$

where  $x^2B(x) = \int_0^x sf(s)ds$ ,  $C(x) = g(x) - xB^2(x)$ .

Let  $\tilde{g}(x) = g(x) - g'(0)x$ ,  $\tilde{f}(x) = \int_0^x sf(s)ds$ , and define

$$\sigma(x) = 2x^2f(x)\tilde{f}(x) - 4\tilde{f}^2(x) + x^3\tilde{g}(x) - x^4\tilde{g}'(x).$$

He obtained the local results as follows.

Let  $f, g \in C^1(a, b)$  for some  $a < 0 < b$ , and the origin be a center. If  $xC(x) > 0$  in a punctured neighborhood of the origin, then, in a neighborhood of the origin,

- (a) if  $\sigma(x) \leq 0$ , then the period function is decreasing;
- (b) if  $\sigma(x) \equiv 0$ , then the period function is a constant;
- (c) if  $\sigma(x) \geq 0$ , then the period function is increasing.

- (12) Cima, Gasull, Mañosa and Mañosas [34] computed the  $m$ -th period constants  $\{T_{2m}\}$ , also the  $m$ -th Liapunov quantities  $\{v_{2m+1}\}$  in a new way. The former is used to study the period function just like the latter is used to study the focus-center problem. In particular, they obtained them for classical Liénard systems. The study of the period function for these systems is continued in [37, 42].

Cima, Gasull and Mañosas [36] gave some properties of the period constants.

In [42], De Maesschalck and Dumortier studied the polynomial Liénard equation (2.6). They supposed that  $f(0) = g(0) = 0$  and the degree of the polynomial  $f$  is  $n$ , which follows that the polynomial  $F(x) = \int_0^x f(x)dx$  has degree  $n + 1$ . They proved that if  $n$  is an odd integer, then the equivalent system

$$\dot{x} = y - F(x), \quad \dot{y} = -x \quad (2.7)$$

with a center at the origin has at least  $n - 1$  critical periods. They also proposed a conjecture that this number is the exact upperbound, which is put in Section 6.

- (13) García-Saldaña and Gasull [53] used the harmonic balance method (it is a different view point of the period function from the traditional) to study the period function for several families of potential non-isochronous systems.
- (14) Chen, Wang and Zhang [17] discussed the problem of reachability of the maximal number of critical periods bifurcated from a weak center with independence of period quantities (like the period constants in point (12) above), and gave a method to judge the reachability without the independence. Chen, Romanovski and Zhang [15] discussed bifurcation of critical periods in an  $m$ -th degree time-reversible system, which is a perturbation of an  $n$ -th degree homogeneous vector field with a rigidly isochronous center at the origin, and reduced the problem of critical periods to finding zeros of a judging function. This procedure gives not only the number of critical periods bifurcating from the period annulus but also the location of these critical periods. Applying this procedure to the case  $n = m = 2$  they determined the maximum number of critical periods and their locations; to the case  $n = m = 3$  they investigated the bifurcation of critical periods up to the first order in  $\varepsilon$  and obtained the expression of the second period-bifurcation function when the first one vanishes.

### 3 Isochronous centers and related topics

- (1) A center point  $p$  of a planar analytic vector field  $X$  in an annulus  $\mathcal{P}$  is an isochronous center if and only if one of the following assertions holds:
- (a) There exists a smooth change of coordinates in a neighborhood of  $p$  that linearizes  $X$  (a classical result of Poincaré). See for instance Theorem 3.3 of [8] or [92].
  - (b) There exists a transversal vector field  $U$  in  $\mathcal{P} \setminus \{p\}$ , commuting with  $X$ , i. e., the Lie bracket  $[X, U] = DXU - UX = 0$ . See [50, 100, 107, 110].

Note that in some books or papers the Lie bracket has an opposite sign, i. e.,  $[X, U] = DXU - UX$ , see for instance [2, 107].

- (c) There exist a transversal vector field  $U$  to  $X$  in  $\mathcal{P} \setminus \{p\}$ , and a scalar function  $\mu : \mathcal{P} \rightarrow \mathbb{R}$ , such that  $[X, U] = \mu X$  and

$$\int_0^{T_\gamma} \mu(x(t), y(t)) dt = 0,$$

where  $\gamma = \{\phi(t) = (x(t), y(t)), t \in [0, T_\gamma]\}$  is any periodic orbit of  $X$  in  $\mathcal{P}$ , and  $T_\gamma$  is its period. In this case,  $U$  is called a normalizer of  $X$ . See [50, 51].

- (d) There exist a transversal vector field  $U$  to  $X$  in  $\mathcal{P} \setminus \{p\}$ , such that

$$\int_0^{T_\gamma} \alpha((x(t), y(t))) e^{-\int_0^t \beta(x(s), y(s)) ds} dt = 0,$$

where  $\alpha$  and  $\beta$  are given by the expression  $[X, U] = \alpha X + \beta U$ ,  $\gamma = \{\phi(t) = (x(t), y(t)), t \in [0, T_\gamma]\}$  is any periodic orbit of  $X$  in  $\mathcal{P} \setminus \{p\}$  and  $T_\gamma$  is its period. See [61].

- (e) There exist a transversal vector field  $U$  to  $X$  in  $\mathcal{P} \setminus \{p\}$  and a scalar function  $\beta$ , such that  $[X, U] = \beta U$ . See [61].

- (2) Mañosas and Villadelprat proved (Theorem B of [87]) that the Hamiltonian system (2.1) has an isochronous center of period  $2\pi$  at the origin if and only if

$$H(x, y) = \frac{g_1(x, y)^2 + g_2(x, y)^2}{2},$$

where  $(x, y) \rightarrow (g_1(x, y), g_2(x, y))$  is an analytic canonical mapping with  $g_1(0, 0) = g_2(0, 0) = 0$ .

The canonical mapping means a mapping such that the determinant of its Jacobian is equal to one at any point (i.e., it is area-preserving).

- (3) Freire, Gasull and Guillamon [50] gave the following result.

Suppose that a vector field  $X$  has an isochronous center of period  $T_0$  in  $\mathcal{P}$ . Consider a vector field  $U$  transversal to  $X$  such that  $[X, U] = 0$ . Let  $\gamma(t) := \{\varphi(t; \psi(h)) \in \mathbb{R}^2, t \in [0, T_0]\}$  be the set of periodic orbits of  $X$  in  $\mathcal{P}$  parameterized by the time flow of  $U$ . Consider the family of vector fields  $X_\varepsilon = X + \varepsilon Y$  having also a center; write  $Y$  as  $Y = aX + bU$  and denote by  $\gamma_\varepsilon(h)$  a generic closed orbit of  $X_\varepsilon$  passing through  $\psi(h)$ . The following statements hold:

- (a) The period function associated to  $\gamma_\varepsilon(h)$  is

$$T_{\gamma_\varepsilon}(h) = T_0 + \varepsilon T_1(h) + O(\varepsilon^2),$$

where

$$T_1(h) = - \int_0^{T_0} a(\varphi(t; \psi(h))) dt.$$

- (b) The derivative of  $T_1$  with respect to  $h$  is

$$T_1'(h) = - \int_0^{T_0} \nabla a(x) \cdot U(x)|_{\{x=\varphi(t; \psi(h))\}} dt. \quad (3.1)$$



- (c) If  $h^*$  is a simple zero of  $T'_1(h)$ , then for small  $\varepsilon$  there is exactly one critical period of  $X_\varepsilon$  close to  $h^*$  which tends to  $h^*$  as  $\varepsilon \rightarrow 0$ .

They also gave the following example in [50]. Consider the system

$$\dot{x} = -y, \quad \dot{y} = x + \varepsilon G'(x). \quad (3.2)$$

Then, for sufficiently small  $\varepsilon$  the zeros of

$$I(s) = \int_0^{2\pi} \frac{x(xG''(x) - G'(x))}{x^2 + y^2} \Big|_{x=s \cos t, y=s \sin t} dt$$

give rise to critical periods of (3.2). Moreover, if  $G'(x)$  is a polynomial of degree  $n$  vanishing at zero, then the maximum number of simple zeros of  $I(s)$  is  $[(n-3)/2]$ , i. e., at most  $[(n-3)/2]$  critical periods bifurcate from the closed orbits of  $(3.2)_{\varepsilon=0}$  in any fixed compact set in the annulus region.

In fact, letting  $X = (-y, x)$ ,  $U = (x, y)$  and  $Y = (0, G'(x))$ , one has  $a = \frac{xG'(x)}{x^2+y^2}$ ,  $b = \frac{yG'(x)}{x^2+y^2}$ ,  $\psi(h) = (e^h, 0)$ ,  $\varphi(t; \psi(h)) = (e^h \cos t, e^h \sin t)$ . Then by changing  $e^h$  to  $s$ , one gets the result. We will use this example later.

- (4) By using the result in Section 3(1)(e) Gasull and Yu [61] obtained a similar result as above, and changed (3.1) to

$$T'_1(h) = - \int_0^{T_0} \nabla a(\mathbf{x}(t, h)) \cdot U(\mathbf{x}(t, h)) e^{-\int_0^t \beta(\mathbf{x}(s, h)) ds} dt, \quad (3.3)$$

where  $[X, U] = \beta U$ ,  $\mathbf{x}(t, h) = \mathbf{x}(t, \mathbf{y}(h, q))$ ,  $\mathbf{y}(h, q)$  denotes the flow of  $U$  satisfying  $\mathbf{y}(0, q) = q$ , and the other notations are the same as in Section 3(3).

- (5) Lukashevich's theorem [4, 86] asserts that each center of system (2.2), whose right-hand sides satisfy the Cauchy-Riemann condition

$$P_x = Q_y, \quad P_y = -Q_x, \quad (3.4)$$

is isochronous. We note that system (2.2), denoted by  $X$ , commutes with the orthogonal system

$$\frac{dx}{dt} = Q(x, y), \quad \frac{dy}{dt} = -P(x, y),$$

denoted by  $Y$ . In fact, by Section 3(1)(b) and condition (3.4) we have

$$[X, Y] = \begin{pmatrix} Q_x & Q_y \\ -P_x & -P_y \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} - \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} \begin{pmatrix} Q \\ -P \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where  $P^2 + Q^2 \neq 0$ . See also [110].

- (6) Chavarriga and Sabatini wrote a long survey article about isochronous centers, see [8] and references therein. To discuss the problem of isochronous centers, one needs first to find the full classification

of centers for a given class of systems, this was completely done only for quadratic systems by Dulac [45], Kapteyn [75], and Frommer [52] (see [3], or [77] and the references therein). For cubic systems it was done only for linear part plus third degree nonlinearity by Sibirskii [105]. For Kukles system it was done only for a “reduced” form. Hence a complete list of isochronous centers was given only for these classes of systems. We will briefly introduce them below, as well as some other results. Note that the center conditions for full Kukles system, formulated by Kukles [76] in 1944, are incomplete. This was first found by Jin and Wang [74], see also [32, 82, 83]. The center conditions for Kukles system were obtained only for its reduced form by Christopher and Lloyd [32]. We will explain the meaning of “reduced” in point (9).

- (7) A complete result for quadratic systems having an isochronous center was obtained by Loud [84], using [106]. Several papers mentioned this result by using different classifications, see for instance [8, 22, 23, 92]. For convenience of later statements, we use the classification by Chicone and Jacobs [23]. The authors of [23] transformed a quadratic system with an isochronous center at the origin to the following Loud’s normal form:

$$\dot{x} = -y + Bxy, \quad \dot{y} = x + Dx^2 + Fy^2. \quad (3.5)$$

The origin is an isochronous center of system (3.5), if and only if either it is linear, i.e., the coefficients satisfy

$$\mathcal{S}_0 : \quad B = D = F = 0;$$

or it is nonlinear and the coefficients satisfy  $B = 1$  and one of the following

$$\mathcal{S}_1 : \quad (D, F) = \left(-\frac{1}{2}, \frac{1}{2}\right);$$

$$\mathcal{S}_2 : \quad (D, F) = (0, 1);$$

$$\mathcal{S}_3 : \quad (D, F) = \left(0, \frac{1}{4}\right);$$

$$\mathcal{S}_4 : \quad (D, F) = \left(-\frac{1}{2}, 2\right).$$

Note that if  $B \neq 0$  in (3.5), then we can make scaling  $(x, y) = (\frac{u}{B}, \frac{v}{B})$ , and the equation for  $(u, v)$  has the same form of (3.5) with  $B = 1$ , and  $(D, F)$  being replaced by  $(\frac{D}{B}, \frac{F}{B})$ .

The phase portraits in the Poincaré disks for these systems can be found in [92].

By using the result in Section 3(4) and the formula (3.3), Gasull and Yu [61] studied the number of critical periods by perturbing the quadratic isochronous centers shown above. Namely, for the system

$$\dot{x} = -y + xy, \quad \dot{y} = x + (D + \varepsilon D_1)x^2 + (F + \varepsilon F_1)y^2, \quad (3.6)$$

where  $D^2 + F^2 \neq 0$ , they obtained the following result.

Fix a compact set  $K$  in the region filled by the periodic orbits of  $(3.6)|_{\varepsilon=0}$ . Then for  $|\varepsilon|$  small enough, up to first order in  $\varepsilon$ , at most one critical period bifurcates from the periodic orbits of system (3.6) contained in  $K$ . Moreover,

- (a) for  $\mathcal{S}_1$ , no critical periods appear. Furthermore for  $|\varepsilon|$  small enough, the period is increasing (decreasing) as the closed orbit runs away from the center at the origin if  $\varepsilon(D_1 + F_1) < 0$  ( $\varepsilon(D_1 + F_1) > 0$ );
- (b) for  $\mathcal{S}_2$ , the critical period can appear if and only if  $-\frac{1}{3} < \frac{D_1}{F_1} < 0$ ;
- (c) for  $\mathcal{S}_3$ , the critical period can appear if and only if  $0 < \frac{D_1}{F_1} < 2$ ;
- (d) for  $\mathcal{S}_4$ , the critical period can appear if and only if  $-\frac{3}{2} < \frac{D_1}{F_1} < 0$ .

We denote by  $\mathcal{C}_2$  the subset of quadratic systems with a center at the origin, and by  $\mathcal{P}$  the related period annulus. In Theorem B of [66] Grau and Villadelprat generalized the above results in [61] to arbitrary analytic in  $\varepsilon$ , that is, if perturb  $\mathcal{S}_2$ ,  $\mathcal{S}_3$  or  $\mathcal{S}_4$  in  $\mathcal{C}_2$  then at most one critical period bifurcates from  $\mathcal{P}$ . Moreover, for each  $k = 0, 1$ , there are perturbations that give rise to exactly  $k$  critical periods bifurcating from  $\mathcal{P}$ .

Remark that in [23] the authors proved that making quadratic perturbations from the classes  $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$  can make at most, respectively, 3, 1, 2, 2, 2 limit cycles locally around the origin. Iliev made a correction that from the class  $\mathcal{S}_1$  the number should be also at most 2, see page 153 of [71].

- (8) A result for a cubic system, with linear part and third degree nonlinearity, having an isochronous center was obtained by Pleshkan [94]. Several papers mentioned this result by using different classifications, see for instance [8, 92]. Here we use the classification by Mardesic, Rousseau and Toni [92].

Consider the systems

$$\dot{x} = -y + P_3(x, y), \quad \dot{y} = x + Q_3(x, y), \quad (3.7)$$

where  $P_3$  and  $Q_3$  are homogenous cubic polynomials, and the origin is a center.

The origin is an isochronous center of system (3.7), if and only if either it is linear, i.e., the coefficients satisfy

$$\mathcal{S}_0^* : P_3(x, y) \equiv 0, \quad Q_3(x, y) \equiv 0;$$

or it is nonlinear and the system can be brought to one of the following systems through a linear change of coordinates and a scaling of time:

$$\mathcal{S}_1^* : \dot{x} = -y(1 + 3x^2 - y^2), \quad \dot{y} = x(1 + x^2 - 3y^2);$$

$$\mathcal{S}_2^* : \dot{x} = -y(1 - x^2), \quad \dot{y} = x(1 + y^2);$$

$$\mathcal{S}_3^* : \dot{x} = -y(1 - 3x^2), \quad \dot{y} = x(1 - 2x^2 + 9y^2);$$

$$\tilde{\mathcal{S}}_3^* : \dot{x} = -y(1 + 3x^2), \quad \dot{y} = x(1 + 2x^2 - 9y^2).$$

The phase portraits in the Poincaré disks for these systems can be found in [92].

We denote by  $\mathcal{C}_3$  the subset of cubic systems with form (3.7) and with a center at the origin, and by  $\mathcal{P}$  the related period annulus.

In Theorem A of [66] Grau and Villadelprat proved that if perturb  $\mathcal{S}_1^*$ ,  $\mathcal{S}_2^*$ ,  $\mathcal{S}_3^*$ , or  $\tilde{\mathcal{S}}_3^*$ , inside  $\mathcal{C}_3$ , then at most two critical periods bifurcate from  $\mathcal{P}$ . Moreover, for each  $k = 0, 1, 2$ , there are perturbations that give rise to exactly  $k$  critical periods bifurcating from  $\mathcal{P}$ .

(9) Consider the Kukles system [76]

$$\dot{x} = -y, \quad \dot{y} = x + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3. \quad (3.8)$$

If  $a_7 = 0$ , this system is called a reduced Kukles system, which was considered by several authors, see [29, 32, 74, 92, 99]. We use the statement in [92] as follows.

The origin is an isochronous center of system (3.8) with  $a_7 = 0$ , if and only if either it is linear, or it is nonlinear and can be brought to the following form through rescaling of  $(x, y)$  and time  $t$ :

$$\dot{x} = -y, \quad \dot{y} = x + 3xy + x^3.$$

The phase portrait in the Poincaré disk for this system can be found in [92].

(10) Rousseau and Toni [98] studied the bifurcation of critical periods of periodic orbits in the neighborhood of a non-degenerate center of the vector field (3.7). They proved that at most three local critical periods bifurcate from a weak linear centre of finite order or from the linear isochrone, and at most two local critical periods from the nonlinear isochrone. Moreover, in both cases, there are perturbations with the maximum number of critical periods. The definition of weak center and its order can be found in pages 439–440 of [22].

Zou, Chen and Zhang [121] studied the local bifurcation of critical periods near a non-degenerate center of the cubic Liénard equation with cubic damping and proved that at most two local critical periods can be produced from either a weak center of finite order or the linear isochronous center, and that at most one local critical period can be produced from nonlinear isochronous centers.

Chen, Huang, Romanovski and Zhang [12] obtained the linearizability and local bifurcation of critical periods for a cubic Kolmogorov system. They used the Darboux method to give explicit linearizing transformations for isochronous centers. Investigating the finite generation for the ideal of all period constants, which are the polynomial form in six parameters, they proved that at most two critical periods can be bifurcated from the interior equilibrium if it is an isochronous center, and the maximum number of critical periods is reachable.

(11) Chavarriga, Giné and García [7] studied a planar system with linear part of center type and nonlinear part given by homogeneous polynomials of fourth degree, and first found some necessary conditions for such isochronous center in polar coordinates.

The necessary and sufficient conditions of isochronous center for systems with linear part and homogeneous polynomials of fifth degree were obtained in [96].

Huang, Chen and Romanovski [70] described a method for studying the center and isochronicity problems for a class of differential systems in the form of linear center perturbed by homogeneous series of degree  $2d + m$  where  $d$  is a nonnegative real number and  $m$  is a positive integer. As an application, the authors classified the centers and isochronous centers for a particular case when  $m = 5$ .

- (12) Cima, Mañosas and Villadelprat [38] considered the Hamiltonian system (2.1) associate to  $H(x, y) = A(x) + B(x)y + C(x)y^2$ , and classified the cubic polynomial Hamiltonian isochronous centers and gave examples of nontrivial and nonglobal polynomial Hamiltonian isochronous centers. For a study of this kind of Hamiltonian functions with more examples, see also the papers by Freire, Gasull and Guillamon [51], and by Garijo and Villadelprat [56].

- (13) Jarque and Villadelprat [73] studied the Hamiltonian system associated to  $H(x, y) = A(x) + B(x)y + C(x)y^2 + D(x)y^3$ , and proved that every center of a polynomial Hamiltonian system of degree 4 (that is, with its homogeneous part of degree 4 not identically zero) is nonisochronous. They also raised a question: is there a planar polynomial Hamiltonian system of even degree which has an isochronous center?

Chen, Romannovski and Zhang [14] gave a criterion for nonisochronicity of the center at the origin of planar polynomial Hamiltonian systems, and answered a weak version of the above question, proving that there is no planar polynomial Hamiltonian system with only even degree nonlinearities having an isochronous center at the origin.

Later, Cresson and Palafox [40] used the correction of a vector field to characterize isochronicity and explicit computations of this quantity for polynomial vector fields, they were able to describe a very large class of nonisochronous Hamiltonian systems of even arbitrarily large degree, and generalized the result in [14].

Wang, Chen and Zhang [113] also studied the above question, and gave a partial answer.

- (14) Devlin [44] gave an example

$$\dot{x} = y + x^4 - 4x^2y^2 - y^4, \quad \dot{y} = -x + 4x^3y,$$

with an isochronous center at  $(0, 0)$  and two nonisochronous centers at  $(\pm 2^{-\frac{2}{3}}, 2^{-\frac{2}{3}})$ . This shows that an isochronous center and a nonisochronous center can coexist in the same system. The phase portrait of this system is shown in Figure 1 of this paper.

- (15) Suppose  $H(x, y) = F(x) + G(y)$  and the origin is a non-degenerate center of  $X_H$ ,  $T(h)$  denotes the period function as above. Cima, Gasull and Mañosas [35] solved the inverse problem of characterizing all systems with a given function  $T(h)$ , characterized the limiting behavior of  $T$  at infinity when the origin is a global center and applied this result to prove, among other results, that there are no nonlinear polynomial isochronous centers in this family.

- (16) Christopher and Devlin [30] studied the isochronicity problem for the Liénard equation (2.6), or the equivalent form

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y, \quad (3.9)$$

where  $f(0) = g(0) = 0$ . They gave necessary and sufficient conditions for such a system with an isochronous center, if  $f$  and  $g$  are analytic or polynomials, respectively.

If  $f$  and  $g$  are polynomials, and their degree is 34 or less, then any system (3.9) with an isochronous center satisfies

$$g(x) = x + \frac{1}{x^3} \left( \int_0^x s f(s) ds \right)^2.$$

See Theorem 3 of [30], where the authors also made a conjecture for degree bigger than 34. We put this conjecture in Section 6. Note that this form of conclusion was first identified by Sabatini in [102] with  $f(x)$  odd.

Chouikha, Romanovski, and Chen [25] gave a method for studying isochronous oscillations in some systems of ODE reducible to the equation

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0.$$

It is applied to obtain the necessary and sufficient conditions for isochronicity of a cubic two-dimensional autonomous system depending on six parameters. For all isochronous systems in this family the Urabe function is explicitly constructed.

- (17) Gavrilov [63] studied the isochronicity problem from complex point of view. He obtained that if the complex Hamiltonian function  $H$  is a non-degenerate semi-weighted homogeneous polynomial, then it cannot have an isochronous Morse critical point, unless the associate Hamiltonian system is linear. He gave a topological obstruction for isochronicity and gave several examples of ‘non-trivial’ complex Hamiltonians with isochronous Morse critical points.
- (18) Chen and Romanovski [13] presented the necessary and sufficient conditions for linearizability of the planar time-reversible cubic complex system

$$\dot{x} = x + P(x, y), \dot{y} = -y + Q(x, y), \quad x, y \in \mathbb{C}, t \in \mathbb{R}.$$

From these conditions, the necessary and sufficient conditions for the origin to be an isochronous center of the time-reversible cubic real system

$$\dot{u} = -v + F(u, v), \dot{v} = u + G(u, v), \quad u, v, t \in \mathbb{R}$$

can be obtained. Thus, the isochronous center problem of time-reversible cubic systems is solved completely.

Similarly, [16] obtained the necessary and sufficient conditions for the origin to be an isochronous center of the planar time-reversible homogeneous quartic polynomials. In the same manner, [11] gave the conditions of isochronous center for a time-reversible quartic-like system.

## 4 Period functions related to elliptic and hyperelliptic hamiltonians

A special case of (2.1) or (2.4) is that

$$H(x, y) = \frac{y^2}{2} + P_n(x), \quad (4.1)$$

where  $P_n$  is a polynomial in  $x$  of degree  $n$ .

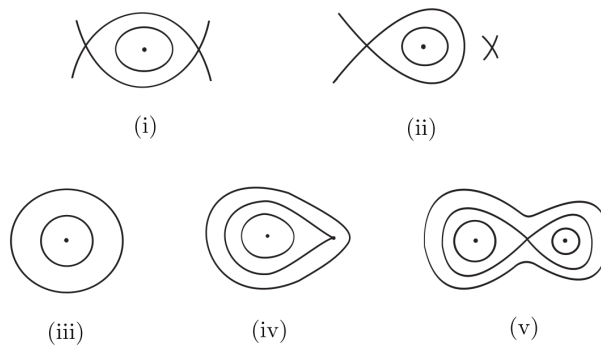
If  $n = 2$ , then the corresponding system (2.1) is linear, and the annulus related to a center, if exists, is isochronous. The function (4.1) is called elliptic Hamiltonian if  $n = 3, 4$ , and hyperelliptic Hamiltonian if  $n \geq 5$ .

### 4.1 The case $n = 3$

Chow and Sanders [27] found that the period function is monotone by using the Picard-Fuchs equations. See also [80]. For the Picard-Fuchs equations, see [41].

### 4.2 The case $n = 4$

In the same paper [27] the authors made a conclusion that the period function has at most three critical points. Later, Gavrilov [62] improved the result that the period function has at most one critical point and this happens only in the global center case, see case (iii) of Figure 1. In this case the system has only one real singularity and a pair of complex singularities. Figure 1 shows all possible phase portraits of the corresponding system associated to the elliptic Hamiltonian of degree 4. Except the case (iii), the period function is monotone in each annulus. Note that there are two annuli with a cuspidal loop related to a double real singularity in case (iv), and three annuli in case (v).



**Figure 1** Classification of level curves of elliptic Hamiltonian of degree 4

### 4.3 The case $n = 5$ and all singularities of $X_H$ are real

As shown by Gavrilov and Iliev [64] that if the annulus (annuli) is (are) located in a compact region, then  $X_H$  has at least two real singularities, and by a suitable change of coordinates the Hamiltonian takes

the following form

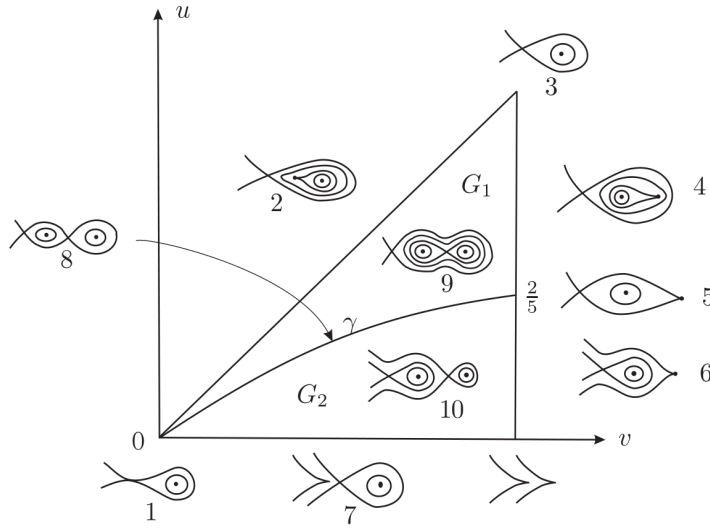
$$H(x, y) = \frac{y^2}{2} + \frac{1}{5}x^5 - \frac{1+u+v}{4}x^4 + \frac{u+v+uv}{3}x^3 - \frac{uv}{2}x^2,$$

where  $u$  and  $v$  are real constants,  $0 \leq u \leq v \leq 1$ , and the corresponding vector field  $X_H$  is

$$\dot{x} = -y, \quad \dot{y} = x(x-u)(x-v)(x-1). \quad (4.2)$$

If  $n = 5$  in (4.1) and all singularities of  $X_H$  are real and  $X_H$  has only one singularity, then this singularity must be quadruple and the system has no closed orbits. Hence, if we study the period function under the condition that all singularities of  $X_H$  are real, then we only need to consider (4.2).

Systems (4.2) has 10 topologically different phase portraits, containing at least one period annulus, shown in Figure 2.



**Figure 2** Classification of level curves of hyperelliptic Hamiltonian of degree 5 with real critical points

1.  $(u, v) = (0, 0)$ . A period annulus surrounds the center  $(x, y) = (1, 0)$  and terminates at the homoclinic loop connecting the degenerate saddle  $(x, y) = (0, 0)$ ;
2.  $\{0 < u = v < 1\}$ . One period annulus surrounds the center  $(x, y) = (1, 0)$  and terminates at the cuspidal loop connecting the degenerated saddle-node  $(x, y) = (u, 0)$ , and another period annulus surrounds the cuspidal loop and terminates at the homoclinic loop connecting the saddle  $(x, y) = (0, 0)$ ;
3.  $(u, v) = (1, 1)$ . A period annulus surrounds the degenerate center  $(x, y) = (1, 0)$  (with multiplicity 3) and terminates at the homoclinic loop connecting the saddle  $(x, y) = (0, 0)$ ;
4.  $\{\frac{2}{5} < u < 1 = v\}$ . One period annulus surrounds the center  $(x, y) = (u, 0)$  and terminates at the cuspidal loop connecting the saddle  $(x, y) = (1, 0)$ , and another period annulus surrounds the cuspidal loop and terminates at the homoclinic loop connecting the saddle  $(x, y) = (0, 0)$ ;



5.  $(u, v) = (\frac{2}{5}, 1)$ . A period annulus surrounds the center  $(x, y) = (u, 0)$  and terminates at the cuspidal loop connecting the saddle-node  $(x, y) = (0, 0)$  and the saddle-node  $(x, y) = (1, 0)$ ;
6.  $\{0 < u < \frac{2}{5}, v = 1\}$ . A period annulus surrounds the center  $(x, y) = (u, 0)$  and terminates at the homoclinic loop connecting the saddle-node  $(x, y) = (0, 0)$ ;
7.  $\{u = 0 < v < 1\}$ . A period annulus surrounds the center  $(x, y) = (1, 0)$  and terminates at the homoclinic loop connecting the saddle  $(x, y) = (v, 0)$ ;
8. Along curve  $\gamma$  defined by  $\{u = \frac{v(5-3v)}{5(2-v)}, 0 < v < 1\}$ , one period annulus surrounds the center  $(x, y) = (1, 0)$  and terminates at the homoclinic loop connecting the saddle  $(x, y) = (v, 0)$ , another period annulus surrounds the center  $(x, y) = (u, 0)$  and terminates at the heteroclinic loop connecting two saddles  $(x, y) = (0, 0)$  and  $(v, 0)$ ;
9. In the open region  $G_1$  surrounded by  $\{u = v\}$ ,  $\{v = 1\}$  and the curve  $\gamma$ , two period annuli surround the centers  $(x, y) = (1, 0)$  and  $(u, 0)$  respectively, and terminate at the eighth loop in the figure connecting the saddle  $(x, y) = (v, 0)$ , and the third period annulus surrounds this eighth loop and terminates at the homoclinic loop connecting the saddle  $(x, y) = (0, 0)$ ;
10. In the open region  $G_2$  surrounded by  $\{u = 0\}$ ,  $\{v = 1\}$  and the curve  $\gamma$ , one period annulus surrounds the center  $(x, y) = (u, 0)$  and terminates at the homoclinic loop connecting the saddle  $(x, y) = (0, 0)$ , and another period annulus surrounds the center  $(x, y) = (1, 0)$  and terminates at the homoclinic loop connecting the saddle  $(x, y) = (v, 0)$ .

It is easy to see that in cases 2, 3, 4, and 9 there is a period annulus which surrounds three singularities, taking into account their multiplicities.

Li and Lu [78] proved that if the period annulus surrounds only one non-degenerate singularity, then the period function is monotonically increasing; otherwise (i.e., the period annulus surrounds the three singularities, taking into account the multiplicities), the period function has exactly one critical point.

To prove the above result, the authors of [78] provided a criterion to determine the convexity of the period function for a class of planar Hamiltonian systems, and combined with other techniques including symbolic computations, to deal with the non-convexity of the period orbit.

Note that if the vector field  $X_H$  has a pair of complex singularities, then the period annulus surrounds a non-degenerate center and terminates at a homoclinic loop connecting to a hyperbolic saddle. In this case the period function can have at least two critical periods.

Remark that by using the criteria of [28], in an early paper Wang [112] considered the period function of the system

$$x'' - x^2(x - \alpha)(x - 1) = 0,$$

where the parameter  $\alpha \in [0, 1)$ . It can be rewritten to the system

$$\dot{x} = -y, \quad \dot{y} = -x^2(x - \alpha)(x - 1). \quad (4.3)$$

If we make the transformation of coordinates  $x = 1 - \tilde{x}$ ,  $y = -\tilde{y}$ , and change  $(\tilde{x}, \tilde{y})$  back to  $(x, y)$ , then system (4.3) becomes

$$\dot{x} = -y, \quad \dot{y} = x[x - (1 - \alpha)](x - 1)^2.$$

This is a special case of (4.2) with  $\{u = 1 - \alpha, v = 1\}$  ( $\alpha \in [0, 1)$ ), and the critical value  $\alpha = \frac{3}{5}$  in [112] corresponds to the critical value  $\{u = \frac{2}{5}, v = 1\}$  in Figure 2. Hence the results in [112] are partial cases in [78]. The methods in these two papers are different.

Yang and Zeng [115] generalized the result in [78], they investigated the cases of  $n > 2$  of (4.1) and supposed that all singularities of the corresponding system are real. Under certain conditions they obtained the behavior of the period function, and that at most one critical period can appear. Their proof was purely analytic.

#### 4.4 Some special cases of $n \geq 5$

- (1) Chicone and Jacobs (see Theorem 4.4 of [22]) considered the number of critical periods, appearing near the center locally, for  $H = \frac{y^2}{2} + V_m(x)$ , where

$$V_m(x) = \frac{1}{2}x^2 + \lambda_3x^3 + \cdots + \lambda_mx^m, \quad m = 5, 6.$$

They proved that for  $V_6$  the system can have at most 4 critical periods bifurcating from the center at origin, and there are perturbations which produce  $k$  critical periods for  $k \leq 3$ ; for  $V_5$  at most 2 critical periods bifurcating from the center, and there are perturbations which produce 2 critical periods.

They also considered the case when  $V(x)$  is an even function, and proved that if

$$V(x) = \frac{1}{2}x^2 + \lambda_4x^4 + \cdots + \lambda_nx^{2n}, \quad \lambda_n \neq 0,$$

then the system can have at most  $n - 2$  local critical periods which bifurcate from the origin.

- (2) Mañosas and Villadelprat [88] considered a special hyperelliptic case of  $n = 6$ , namely,

$$H = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{a}{4}x^4 + \frac{b}{6}x^6, \quad b \neq 0.$$

By using the Picard-Fuchs equation they obtained that the period function has at most one critical point, although in some cases there are four period annuli. More precisely, for the corresponding system the following hold:

- (a) In the case that  $b < 0$ , there is only the period annulus of the center at the origin, which is bounded. If  $a \leq 0$  then its period function is monotonous increasing, and if  $a > 0$  then it has one simple critical period, which is a minimum.
- (b) In the case that  $b > 0$  and  $a > -2\sqrt{b}$ , there is only the period annulus of the center at the origin, which is global. If  $a \geq 0$  then its period function is monotonous decreasing, and if  $a \in (-2\sqrt{b}, 0)$  then it has one simple critical period, which is a maximum.

- (c) In the case  $b > 0$  and  $a < -2\sqrt{b}$ , there are four period annuli. The period functions associated to the three bounded period annuli are monotonous increasing and the one associated to the unbounded one is monotonous decreasing.
- (d) In the case  $b > 0$  and  $a = -2\sqrt{b}$ , there are two period annuli. The period function associated to the bounded period annulus is monotonous increasing and the one associated to the unbounded one is monotonous decreasing.

The authors also gave the bifurcation diagram of the period function of the center at the origin.

- (3) By using their criterion, shown in Section 2(10), Mañosas and Villadelprat [90] gave an example with

$$H = \frac{y^2}{2} + \frac{x^m}{ax^n + 1},$$

where  $m$  and  $n$  are even natural numbers. This system has a center at the origin, non-degenerate for  $m = 2$  and degenerate for  $m \geq 4$ . Setting

$$\mathcal{B}(m, n) = n^2 + 2mn - 2m^2 - 2n + 4m + 1,$$

they proved that if  $a < 0$  then the period function is monotone decreasing; if  $a > 0$  then the following hold:

- (a) if  $m \leq n$ , then the period function has a unique critical period (a minimum) when  $m \geq 4$ , and it is monotone increasing when  $m = 2$ ;
- (b) if  $m > n$ , then the period function is monotone decreasing in the case  $\mathcal{B}(m, n) < 0$ .
- (4) In the same paper [90] the authors gave another example with

$$H = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{a}{4}x^4 + \frac{1}{6}x^6 + \frac{1}{8}x^8, \quad a < 0.$$

They proved that the corresponding system has a global center at the origin if and only if  $a > a^*$ , where  $a^* \approx -2.610$  is the unique real root of  $4a^3 - a^2 - 18a + 31 = 0$ . Moreover, if  $a \in (a_1, 0)$ , where  $a_1 \approx -0.668$ , then the system has exactly one critical period but its period function is not convex.

## 5 Period functions of quadratic systems

The classification of quadratic integrable systems is as follows, see Schlomiuk [104], Zoladek [120] and Iliev [71].

- (i)  $\dot{z} = -iz - z^2 + 2|z|^2 + (b + ic)\bar{z}^2$ , Hamiltonian  $(Q_3^H)$ ,
- (ii)  $\dot{z} = -iz + az^2 + 2|z|^2 + b\bar{z}^2$ , reversible  $(Q_3^R)$ ,
- (iii)  $\dot{z} = -iz + 4z^2 + 2|z|^2 + (b + ic)\bar{z}^2$ ,  $|b + ic| = 2$ , codimension four  $(Q_4)$ ,

(iv)  $\dot{z} = -i z + z^2 + (b + i c)\bar{z}^2$ , generalized Lotka-Volterra ( $Q_3^{LV}$ ),

where  $a, b$  and  $c$  are complex,  $z = x + i y$  is a complex variable. Note that the Hamiltonian triangle is the intersection of  $Q_3^H \cap Q_3^R \cap Q_3^{LV}$ .

The following results show the behavior of the period function for quadratic integrable systems with period annulus (or annuli). Note that the results are complete for systems belong to  $Q_3^H \cup Q_4$ , and not complete for  $Q_3^{LV} \cup Q_3^R$ . A lot of papers are published, but the remaining cases seem quite difficult.

### 5.1 Complete result for Hamiltonian class $Q_3^H$

Coppel and Gavrilov [39] proved that the period function for the quadratic Hamiltonian systems with a center, including the Hamiltonian triangle, is monotone. They first transformed the quadratic systems  $X \in Q_3^H$  to the potential form (2.4), then found that

$$T'(h) = \frac{1}{2h} \iint_{\sigma(h)} \left( \frac{V(x)}{(V'(x))^2} \right)'' dx dy,$$

where  $\sigma(h)$  is the compact region bounded by  $\gamma(h)$ . At last, they proved that the function  $\frac{V(x)}{(V'(x))^2}$  is well defined for all  $x \in \mathcal{I}$  (even for  $x = 0$ ), and is convex. The author found that their proof is compatible with the criteria of Sections 2(7) and 2(8).

### 5.2 Complete result for codimension four class $Q_4$

Zhao [117] proved that the period functions for the quadratic codimension four systems ( $X \in Q_4$ ) are monotone. The main tool in this paper is the Picard-Fuchs equation and the Riccati equation.

### 5.3 Partial results for $Q_3^{LV} \cup Q_3^R$

- (1) Chicone and Jacobs [22] proved that at most two local critical periods bifurcate from quadratic centers. Here “local” means when the perturbation parameter  $\varepsilon$  tends to zero, the level curves with bifurcating critical periods shrink to the center point.

By the spirit of Bautin [3] to treat the multiple Hopf bifurcation, the authors of [22] studied the structure of the Taylor coefficients of the period function, and determined the bifurcation structure of the critical periods for quadratic systems near the center. Of course, this is a challenging problem.

- (2) By using the Picard-Fuchs equation and the related Riccati equation Zhao [119] proved that for the quadratic integrable system with centers whose almost all orbits lie on cubic planar curves, the corresponding period function has at most one critical point.

Sections 5.1 and 5.2 showed that there is no critical period for  $X \in \{Q_3^H \cup Q_4\}$ , hence the above two results are only for  $Q_3^{LV} \cup Q_3^R$ .

#### 5.4 Partial results for generalized Lotka-Volterra class $Q_3^{LV}$

- (1) Rothe [97], Schaaf [103] and Waldvogel [111] proved independently that the period functions are monotone for classical quadratic Lotka-Volterra systems of the form

$$\dot{u} = \lambda u(1 - v), \quad \dot{v} = -\mu v(1 - u), \quad (5.1)$$

with constants  $\lambda > 0, \mu > 0$ . By using the transformation  $u = e^x, v = e^y$  system (5.1) can be changed to a Hamiltonian system

$$\dot{x} = \lambda(1 - e^y), \quad \dot{y} = -\mu(1 - e^x).$$

Then Schaaf introduced polar coordinates in [103], and used his criterion, given in Section 2(8). He also pointed out that his criterion is related to the conditions in [97], and his approach is very similar to the one of [111], although the approaches in other two papers are quite different.

- (2) Villadelprat [109] proved that the period function is monotone increasing near the outer boundary of the period annulus for general quadratic Lotka-Volterra systems, if it has a center, other than the isochronous case.
- (3) Long, Liu and Wang [85] proved that if the invariant lines of the system in  $Q_3^{LV}$  are real, then the period function is monotone.

#### 5.5 Partial results for reversible class $Q_3^R$

From Proposition 4.4 and equation (4.10) of [104], or from Section 1.4 and equation (5) of [120] we know that by an affine transformation and a constant scaling of time, any reversible quadratic system can be changed to the Loud's normal form

$$\dot{x} = -y + Bxy, \quad \dot{y} = x + Dx^2 + Fy^2. \quad (5.2)$$

Note that this is the same equation of (3.5). See also [84, 108].

By using the methods shown in Section 3(3), Gasull, Guillamon and Villadelprat [59] proved that if  $B = 0$  and  $D^2 + F^2 \neq 0$  in (5.2), then the period function of the center at the origin is monotonous increasing.

If  $B \neq 0$ , then by a scaling one can change to the case of  $B = 1$ . Hence the remaining part is to study

$$\dot{x} = -y + xy, \quad \dot{y} = x + Dx^2 + Fy^2. \quad (5.3)$$

Chicone and Jacobs [22] called (5.3) a dehomogenized Loud's system and found that the most interesting and complicated phenomena of the period function for quadratic centers occur in this family. The study has not completed until now, although many papers in this direction were published. We list some of them below.

- (1) Chicone and Dumortier [20] found that if  $F = 2$  and  $-D \in (\frac{7}{5}, c^*)$ ,  $c^* \approx 1.47$  in (5.3), then the system has at least one critical period. This probably is the first example of quadric systems with non-monotone period function.
- (2) The above result was generalized by Zhao [118]. He obtained the behaviour of the period function for  $(D, F) \in \mathbb{R}^2$  in the straight line  $\{F = 2, D \in (-\infty, +\infty)\}$  of (5.3): the system has two isochronous centers for  $-D = \frac{1}{2}$ , it has only one critical period for  $-D \in (\frac{7}{5}, 2)$ , and its period function is monotone for all other values of  $D$ . The tools of [118] is the Picard-Fuchs equation and the related Riccati equation.
- (3) In paper [91] Mardesic, Marín and Villadelprat determined the behavior of the period function for  $X \in Q_3^R$ , they gave some regions in the parameter plane, where the corresponding system has at least one or two critical periods near the saddle loop, and gave the local bifurcation diagram about this behaviour.

Combining with the results of [22] about the number of critical periods near the center of quadratic systems, which we introduced in Section 5.3(1), the authors also made a conjecture of bifurcation diagram of the period function for system (5.3).

The study was continued by Rojas and Villadelprat [95] and Marín and Villadelprat [93]. In the latter paper there are new results not proved in [91] and a more accurate conjectural bifurcation diagram, see Conjecture 3 in Section 6.

- (4) By using Schaaf's criterium (see Section 2(8)) Villadelprat [108] proved that the period function of (5.3) is monotone for  $(D, F)$  in several regions of  $\mathbb{R}^2$ , see Figures 1 and 2 of [108]. The study was continued by Mañosas and Villadelprat [89] to find more regions for  $(D, F) \in \mathbb{R}^2$  that the period function is monotone or it has critical period(s).
- (5) Chouikha [24] obtained the monotonicity of the period function for parameters in the straight lines  $\{F + 2D = 0\}$  and  $\{F = -1\}$ , and some segments inside  $\{D = -\frac{1}{2}\}$ ,  $\{D = 0\}$ ,  $\{F = 1\}$ , and  $\{F = 2\}$ .

Note that the result of  $\{F = -1\}$  was also obtained independently in the paper [108] and the result of  $\{F = 2\}$  is covered by [118].

## 5.6 Complete results for $Q_3^{LV} \cap Q_3^R$

Recently Li, Li, Liu and Wang [81] proved that the period function for any reversible Lotka-Volterra system is monotone, except an isochronous case.

Using the results in [104], choosing parameters satisfying  $b + d = a = C = 0$  in equation (3.S') of this paper and exchanging the variables  $x$  and  $y$ , we obtain that any system  $X \in Q_3^{LV} \cap Q_3^R$  has the form

$$\dot{x} = -y + Axy, \quad \dot{y} = x - bx^2 + by^2. \quad (5.4)$$

As shown in Section 5.5 that if  $A = 0$  and  $b \neq 0$  in (5.4) then the period function of the system is monotone ([59]); if  $A \neq 0$  then it can be changed to  $A = 1$ . Hence we only need to consider the system (changing  $b$  to  $a$ )

$$\dot{x} = -y + xy, \quad \dot{y} = x - ax^2 + ay^2, \quad (5.5)$$

where  $a \in (-\infty, +\infty)$ .

Note that system (5.5) has a unique center  $(0, 0)$  if  $a \in (-\infty, 0] \cup [1, +\infty)$ , and has two centers  $(0, 0)$  and  $(\frac{1}{a}, 0)$  if  $a \in (0, 1)$ . Moreover, if  $a = \frac{1}{2}$  then both centers  $(0, 0)$  and  $(2, 0)$  of system (5.5) are isochronous, see Section 3(7)  $\mathcal{S}_1$ .

The result in [81] is as follows.

For system (5.5), if  $a \in (-\infty, 0] \cup [1, +\infty)$  then the period function of the only annulus is monotone; if  $a \in (0, 1) \setminus \{\frac{1}{2}\}$ , then the period functions of the two annuli are monotone; if  $a = \frac{1}{2}$ , then both centers are isochronous.

Remark that the result for  $a < 0$  and a part of  $a > 1$  were covered by [108], but the result for  $a \in (0, 1) \setminus \{\frac{1}{2}\}$  was open. The authors of [81] used the criterion in Section 2(10) with some modifications, and combined the analytic computations with a geometric idea and variation of the parameter  $a$ , as well as symbolic computations.

## 6 Conjectures and problems

Related to some results in this article, there are several conjectures and problems below. We give more or less explanations or discussions.

**Conjecture 1** ([30]) If  $f$  and  $g$  are polynomials then the Liénard system (3.9) has an isochronous center at the origin if and only if

$$g(x) = x + \frac{1}{x^3} \left( \int_0^x s f(s) ds \right)^2.$$

See Section 3(16).

**Conjecture 2** ([42]) Suppose that a classical polynomial Liénard system (2.7) has a center at the origin. For any odd integer  $n$  an upperbound for the number of critical periods that the system (2.7) of degree  $n + 1$  can have is given by  $n - 1$ .

See Section 2(12).

**Conjecture 3** In the hyperelliptic case of  $n = 5$ , if the level set  $\{H = h\}$  contains ovals and  $X_H$  has a pair of complex singularities, then the period function has at most two critical points.

If this conjecture is true, then combining with the result in Section 4.3, we can obtain the complete conclusion for the hyperelliptic case of  $n = 5$ . Note that if  $X_H$  has two pairs of complex singularities then the system has no any closed orbit.

If we take  $H = \frac{y^2}{2} + \frac{x^2}{2} + \varepsilon P_n(x)$ , where  $\varepsilon$  is a small parameter and  $P_n$  is a polynomial of  $x$  from degree 3 to  $n$ , then by using the results of equation (3.2), we can get the conclusion that the period function of the corresponding system, up to the first order of  $\varepsilon$ , has at most  $\lfloor \frac{n-2}{2} \rfloor$  critical points. In general we may ask the following question.

**Problem 1** How about the behaviour of the period function for the system associated to a hyperelliptic Hamiltonian of degree  $n > 5$ ?

**Conjecture 4** ([19]) If the period function of any quadratic system has a critical point(s), then it can be transformed to dehomogenized Loud's form (5.3), and has at most two critical periods.

If the first part of this conjecture is true, then combining with the results in [81], we can obtain that the period function for  $X \in Q_3^{LV}$  is either isochronous or monotone. The second part of this conjecture is really a challenging problem, see Section 5.5 and the following conjecture.

**Conjecture 5** ([93]) The bifurcation diagram of the period function for system (5.3) and the behavior of the period function in different regions of  $(D, F) \in \mathbb{R}^2$  are shown in Figure 7 of [93].

This conjecture is based on Conjecture 4 and a series of works concerning the period function by Villadelprat and his co-authors.

**Problem 2** ([73]) Is there a planar polynomial Hamiltonian system of even degree which has an isochronous center?

See Section 3(13) for detailed information about this question.

The behavior of the isochronous centers is related to the famous Jacobian conjecture for  $n = 2$ , see Section 6 of [63] for a statement of this conjecture. We state the problem as shown in [101]. Suppose that  $f(x, y), g(x, y) \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ , and  $f(0, 0) = g(0, 0) = 0$ . Consider the Hamiltonian function in the form

$$H(x, y) = \frac{f(x, y)^2 + g(x, y)^2}{2},$$

the corresponding Hamiltonian systems is

$$\dot{x} = -H_y = -ff_y - gg_y, \quad \dot{y} = H_x = ff_x + gg_x. \quad (6.1)$$

**Problem 3** ([101]) Do there exist Hamiltonian polynomial systems of form (6.1), with canonical mapping  $(x, y) \rightarrow (f, g)$ , having isochronous nonglobal centers?

The Jacobian conjecture in  $\mathbb{R}^2$  is true if and only if the above problem has a negative answer. The canonical mapping means the determinant of its Jacobian is identically equal to one, see Section 3(2).

Let  $\mathcal{H}(n)$  denote the maximal number of limit cycles that planar polynomial systems of degree  $n$  can have. From [33, 67] it is known that

$$\mathcal{H}(n) \geq Kn^2 \ln(n), \quad \text{for some } K > 0.$$

On the other hand, if we denote as  $\mathcal{T}(n)$  the maximal number of critical periods that planar polynomial systems of degree  $n$  can have, then from the results of [60] it is known that

$$\mathcal{T}(n) \geq \frac{1}{4}n^2.$$

This lower bound has been essentially doubled in [6], see Section 2(9). A natural question is:

**Problem 4** ([57]) Is it true that  $\mathcal{T}(n) \geq Cn^2 \ln(n)$  for some constant  $C > 0$ ?

More open problems can be found in Section 3 of [57]. Note that [81] gave an answer to the Problem 16 of [57]. In fact, in the last equation of page 248 of [57] (where  $b \geq 1$ ) if we make change  $x = \tilde{x} + (1+b)$ ,



let  $(\tilde{x}, y, t) = (-bu, \sqrt{b}v, \frac{1}{\sqrt{b}}\tau)$ , and at last change  $(u, v, \tau)$  back to  $(x, y, t)$ , then we obtain (5.5) with  $a = \frac{b}{1+b} \in [\frac{1}{2}, 1)$ , see Section 5.6.

### Acknowledgements

The author thanks Professors Chen Xingwu, Armengol Gasull, Liu Changjian, Joan Torregrosa, Jordi Villadelprat, Xiao Dongmei, Yu Jiang, Zhao Yulin, and the reviewers of this article for their valuable comments and for their suggestions to add more references.

### References

- [1] Álvarez M J, Gasull A, Prohens R. Global behaviour of the period function of the sum of two quasi-homogeneous vector fields[J]. *Journal of Mathematical Analysis & Applications*, 2017, 449(2): 1553–1569.
- [2] Arnold V I. *Mathematical methods of classical mechanics*[M]. New York: Springer-Verlag, 1978.
- [3] Bautin N N. On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type[J]. *Trudy Akad. Nauk SSSR Ser. Matem. Mekh.*, 1952, 30(72): 181–196 (in Russian); *American Mathematical Society Translations*, 1954, 100(1): 397–413.
- [4] Bonorino L P, Brietzke E, Lukaszczyk J P, et al. Properties of the period function for some Hamiltonian systems and homogeneous solutions of a semilinear elliptic equation[J]. *Journal of Differential Equations*, 2005, 214(1): 156–175.
- [5] Buzzi C A, Carvalho Y R, Gasull A. The local period function for Hamiltonian systems with applications[J]. *Journal of Differential Equations*, 2021, 280(4): 590–617.
- [6] Cen X. New lower bound for the number of critical periods for planar polynomial systems Science Direct[J]. *Journal of Differential Equations*, 2021, 271: 480–498.
- [7] Chavarriga J, Giné J, García I A. Isochronous centers of a linear center perturbed by fifth degree homogeneous polynomials[J]. *Journal of Computational and Applied Mathematics*, 1999, 123 : 77–96.
- [8] Sabatini M, Chavarriga J. A survey of isochronous centers[J]. *Qualitative Theory of Dynamical Systems*, 1999, 1: 1–70.
- [9] Chen F, Li C, Llibre J, et al. A unified proof on the weak Hilbert 16th problem for  $n = 2$ [J]. *Journal of Differential Equations*, 2006, 221(2): 309–342.
- [10] Chen G, Li C, Liu C, et al. The cyclicity of period annuli of some classes of reversible quadratic systems[J]. *Discrete & Continuous Dynamical Systems*, 2006, 16: 157–177.
- [11] Chen X, Huang H, Romanovski V G, et al. Linearizability conditions of a time-reversible quartic-like system[J]. *Journal of Mathematical Analysis and Applications*, 2011, 383: 179–189.

- 
- [12] Chen X, Huang W, Romanovski V G, et al. Linearizability and local bifurcation of critical periods in a cubic Kolmogorov system[J]. *Journal of Computational & Applied Mathematics*, 2013, 245(1): 86–96.
- [13] Chen X, Romanovski V G. Linearizability conditions of time- reversible cubic systems[J]. *Journal of Mathematical Analysis & Applications*, 2010, 362(2): 438–449.
- [14] Chen X, Romanovski V G, Zhang W. Non-isochronicity of the center at the origin in polynomial Hamiltonian systems with even degree nonlinearities[J]. *Nonlinear Analysis Theory Methods & Applications*, 2008, 68(9): 2769–2778.
- [15] Chen X, Romanovski V G, Zhang W. Critical periods of perturbations of reversible rigidly isochronous centers[J]. *Journal of Differential Equations*, 2011, 251(6): 1505–1525.
- [16] Chen X, Romanovski V G, Zhang W. Linearizability conditions of time- reversible quartic systems having homogeneous nonlinearities[J]. *Nonlinear Analysis*, 2008, 69(5–6): 1525–1539.
- [17] Chen X, Wang Z, Zhang W. Reachability of maximal number of critical periods without independence[J]. *Journal of Differential Equations*, 2020, 269(11): 9783–9803.
- [18] Chicone C. The monotonicity of the period function for planar Hamiltonian vector fields[J]. *Journal of Differential Equations*, 1987, 69(3): 310–321.
- [19] Chicone C. Review of: “The period function of a Hamiltonian quadratic system” by Coppel W A and Gavrilov L[Z]. *Mathematical Reviews*, MR: 1235199 (94h: 58072), 1994.
- [20] Chicone C, Dumortier F. A quadratic system with a nonmonotonic period function[J]. *Proceedings of the American Mathematical Society*, 1988, 102: 706–710.
- [21] Chicone C, Dumortier F. Finiteness for critical periods of planar analytic vector fields[J]. *Nonlinear Analysis Theory Methods & Applications*, 1993, 20(4): 315–335.
- [22] Chicone C, Jacobs M. Bifurcation of Critical Periods for Plane Vector Fields[J]. *Transactions of the American Mathematical Society*, 1989, 312: 433–486.
- [23] Chicone C, Jacobs M. Bifurcation of limit cycles from quadratic isochrones[J]. *Journal of Differential Equations*, 1991, 91(2): 268–326.
- [24] Chouikha A R. Monotonicity of the period function for some planar differential systems. Part I: Conservative and quadratic systems[J]. *Applicationes Mathematicae*, 2005, 32(3): 305–325.
- [25] Chouikha A R, Romanovski V G, Chen X. Isochronicity of analytic systems via Urabeš criterion[J]. *Journal of Physics A: Mathematical and Theoretical*, 2007, 40(10): 2313–2327.
- [26] Chow S N, Li C, Yi Y. The Cyclicity of Period Annulus of Degenerate Quadratic Hamiltonian System with Elliptic Segment[J]. *Ergodic Theory & Dynamical Systems*, 2002, 22(02): 349–374.
- [27] Chow S N, Sanders J A. On the number of critical points of the period[J]. *Journal of Differential Equations*, 1986, 64(1): 51–66.
- [28] Chow S N, Wang D. On the monotonicity of the period function of some second order equations[J]. *Časopis Pěstování Matematiky*, 1986, 111(1): 14–25.
- [29] Christopher C, Devlin J. Isochronous Centers in Planar Polynomial Systems[J]. *SIAM Journal on Mathematical Analysis*, 1997, 28(1): 162–177.

- 
- [30] Christopher C, Devlin J. On the classification of Liénard systems with amplitude- independent periods[J]. *Journal of Differential Equations*, 2004, 200(1): 1-17.
- [31] Christopher C, Li C. *Limit Cycles of Differential Equations (Advanced Courses in Mathematics CRM Barcelona)*[M]. Birkhäuser Verlag, 2007.
- [32] Christopher C, Lloyd N G. On the paper of Jin and Wang concerning the conditions for a centre in certain cubic systems[J]. *Bulletin of the London Mathematical Society*, 1990, 22: 5-12.
- [33] Christopher C, Lloyd N G. Polynomial Systems: A Lower Bound for the Hilbert Numbers[J]. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 1995, A450: 219-224.
- [34] Cima A, Gasull A, Mañosa V, et al. Algebraic Properties of the Liapunov and Period Constants[J]. *Rocky Mountain Journal of Mathematics*, 1997, 27(2): 471-501.
- [35] Cima A, Gasull A, Mañosas F. Period Function for a Class of Hamiltonian Systems[J]. *Journal of Differential Equations*, 2000, 168(1): 180-199.
- [36] Cima A, Gasull A, Mañosas F. A note on the Lyapunov and Period constants[J]. *Qualitative Theory of Dynamical Systems*, 2020, 19(1): 1-13.
- [37] Cima A, Gasull A, Silva P. On the number of critical periods for planar polynomial systems[J]. *Nonlinear Analysis*, 2008, 69(7): 1889-1903.
- [38] Cima A, Mañosas F, Villadelprat J. Isochrony for several classes of Hamiltonian systems[J]. *Journal of Differential Equations*, 1999, 157: 373-413.
- [39] Coppel W A, Gavrilov L. The period function of a Hamiltonian quadratic system[J]. *Differential and Integral Equations*, 1993, 6: 1337-1365.
- [40] Cresson J, Palafox J. Isochronous centers of polynomial Hamiltonian systems and a conjecture of Jarque and Villadelprat[J]. *Journal of Differential Equations*, 2019, 266: 5713-5747.
- [41] Cushman R, Sanders J A. Codimension two bifurcation with a third order Picard-Fuchs equation[J]. *Journal of Differential Equations*, 1985, 59: 243-256.
- [42] De Maesschalck P, Dumortier F. The period function of classical Liénard equations[J]. *Journal of Differential Equations*, 2007, 233: 380-403.
- [43] De Maesschalck P, Torregrosa J. Limit cycles and critical periods with non-hyperbolic slow-fast systems, Preprint.
- [44] Devlin J. Coexisting Isochronous and Nonisochronous Centres[J]. *Bulletin of the London Mathematical Society*, 1996, 28(5): 495-500.
- [45] Dulac H. Détermination et intégration d'une certaine classe d'équations différentielles ayant pour point singulier un centre[J]. *Bulletin des Sciences Mathématiques*, 1908, 32: 230-252.
- [46] Dumortier F, Li C. Perturbations from an elliptic Hamiltonian of degree four: (I) saddle loop and two saddle cycle[J]. *Journal of Differential Equations*, 2001, 176: 114-157.
- [47] Dumortier F, Li C. Perturbations from an elliptic Hamiltonian of degree four: (II) cuspidal loop[J]. *Journal of Differential Equations*, 2001, 175: 209-243.
- [48] Dumortier F, Li C. Perturbation from an elliptic Hamiltonian of degree four: (III) Global center[J].

- Journal of Differential Equations, 2003, 188: 473–511.
- [49] Dumortier F, Li C. Perturbation from an elliptic Hamiltonian of degree four: (IV) Figure eight-loop[J]. Journal of Differential Equations, 2003, 188: 512–554.
- [50] Freire E, Gasull A, Guillamon A. Period function for perturbed isochronous centres[J]. Qualitative Theory of Dynamical Systems, 2002, 3: 275–284.
- [51] Freire E, Gasull A, Guillamon A. A characterization of isochronous centres in terms of symmetries[J]. European Mathematical Society, 2004, 20: 205–222.
- [52] Frommer M. Über das auftreten von wirbeln und strudeln (geschlossener und spiraliger integralkurven) in der umgebung rationaler unbestimmtheitsstellen[J]. Mathematische Annalen, 1934, 109: 395–424.
- [53] García-Saldaña J D, Gasull A. The period function and the Harmonic Balance Method[J]. Bulletin Des Sciences Mathématiques, 2015, 139(1): 33–60.
- [54] Garijo A, Gasull A, Jarque X. On the period function for a family of complex differential equations[J]. Journal of Differential Equations, 2006, 224(2): 314–331.
- [55] Garijo A, Gasull A, Jarque X. A note on the period function for certain planar vector fields[J]. Journal of Difference Equations & Applications, 2010, 16(5–6): 631–645.
- [56] Garijo A, Villadelprat J. Algebraic and analytical tools for the study of the period function[J]. Journal of Differential Equations, 2014, 257(7): 2464–2484.
- [57] Gasull A. Some open problems in low dimensional dynamical systems[J]. SeMa Journal, 2021, 78: 233–269.
- [58] Gasull A, Guillamon A, Mañosa V, et al. The Period Function for Hamiltonian Systems with Homogeneous Nonlinearities[J]. Journal of Differential Equations, 1997, 139(2): 237–260.
- [59] Gasull A, Guillamon A, Villadelprat J. The period function for second-order quadratic ODEs is monotone[J]. Qualitative Theory of Dynamical Systems, 2004, 4(2): 329–352.
- [60] Gasull A, Liu C, Yang J. On the number of critical periods for planar polynomial systems of arbitrary degree[J]. Journal of Differential Equations, 2010, 249: 684–692.
- [61] Gasull A, Yu J. On the critical periods of perturbed isochronous centers[J]. Journal of Differential Equations, 2008, 244: 696–715.
- [62] Gavrilov L. Remark on the number of critical points of the period[J]. Journal of Differential Equations, 1993, 101: 58–65.
- [63] Gavrilov L. Isochronicity of plane polynomial Hamiltonian systems[J]. Nonlinearity 1997, 10: 433–448.
- [64] Gavrilov L, Iliev I D. Complete hyperelliptic integrals of the first kind and their non-oscillation[J]. American Mathematical Society (AMS), 2003, 356: 1185–1207.
- [65] Grau M, Mañosas F, Villadelprat J. A Chebyshev criterion for Abelian integrals[J]. Transactions of the American Mathematical Society, 2011, 363: 109–129.
- [66] Grau M, Villadelprat J. Bifurcation of critical periods from Pleshkan’s isochrones[J]. Journal of the London Mathematical Society, 2010, 81: 142–160.

- 
- [67] Han M, Li J. Lower bounds for the Hilbert number of polynomial systems[J]. Journal of Differential Equations, 2012, 252: 3278–3304.
- [68] He H, Xiao D. On the global center of planar polynomial differential systems and the related problems[J]. Journal of Applied Analysis and Computation, 2022, 12(3): 1141–1157.
- [69] He H, Llibre J, Xiao D. Planar polynomial Hamiltonian differential systems with global centers[J]. Scientia Sinica Mathematica (in Chinese), 2022, 52: 617–628.
- [70] Huang W, Chen X, Romanovski V G. Linear centers with perturbations of degree  $2d + 5$ [J]. International Journal of Bifurcation & Chaos, 2012, 22: 1250007,1–12.
- [71] Iliev I D. Perturbations of quadratic centers[J]. Bulletin des Sciences Mathématiques, 1998, 122: 107–161.
- [72] Iliev I D, Li C, Yu J. Bifurcations of limit cycles from quadratic non- Hamiltonian systems with two centres and two unbounded heteroclinic loops[J]. Nonlinearity, 2004, 18(1): 305–330.
- [73] Jarque X, Villadelprat J. Nonexistence of Isochronous Centers in Planar Polynomial Hamiltonian Systems of Degree Four[J]. Journal of Differential Equations, 2002, 180: 334–373.
- [74] Jin X, Wang D. On the conditions of Kukles' for the existence of a centre[J]. Bulletin of the London Mathematical Society, 1990, 22: 1–4.
- [75] Kaptegn W. On the centre of the integral curves which satisfy differential equations of the first order and the first degree[J]. Akdemie Von Wetenschappen te Amsterdam, Proceedings of the Section of Science, 1911, 13(2): 1241–1252.
- [76] Kukles I S. Sur quelques cas de distinction entre un foyer et un centre[J]. Doklady Akademii Nauk SSSR, 1944, 42: 208–211.
- [77] Li C. Two problems of planar quadratic systems[J]. Scientia Sinica (Series A), 1982, 12: 1087 – 1096 (in Chinese); 1983, 26(5): 471–481 (in English).
- [78] Li C, Lu K. The period function of hyperelliptic Hamiltonians of degree 5 with real critical points[J]. Nonlinearity, 2008, 21(3): 465–483.
- [79] Li C, Zhang Z. A Criterion for determining the monotonicity of the ratio of two Abelian integrals[J]. Journal of Differential Equations, 1996, 124: 407–424.
- [80] Li C G, Li C. New proofs of monotonicity of period function for cubic elliptic Hamiltonian[J]. Journal of Nonlinear Modeling and Analysis, 2019, 1(3): 301–330.
- [81] Li J, Li C, Liu C, et al. The period function of reversible Lotka- Volterra quadratic centers[J]. Journal of Differential Equations, 2022, 307: 556–579.
- [82] Lloyd N G, Pearson J M. Conditions for a centre and the bifurcation of limit cycles in a class of cubic systems. Bifurcations of planar vector fields[J]. Lecture Notes in Math. 1990, 1455: 230–242.
- [83] Lloyd N G, Pearson J M. Computing centre conditions for certain cubic systems[J]. Journal of Computational and Applied Mathematics, 1992, 40(3): 323–336.
- [84] Loud W S. Behavior of the period of solutions of certain plane autonomous systems near centers[J]. Contributions to Differential Equations, 1964, 3: 21–36.
- [85] Long T, Liu C, Wang S. The period function of quadratic generalized Lotka-Volterra systems

- without complex invariant lines[J]. *Journal of Differential Equations*, 2022, 314: 491–517.
- [86] Lukashevich N A. Isochronicity of a center for certain systems of differential equations[J]. *Differentsial'nye Uravneniya* (in Russian), 1965, 1(5): 295–302.
- [87] Mañosas F, Villadelprat J. Area-preserving normalizations for centers of planar Hamiltonian systems[J]. *Journal of Differential Equations*, 2002, 179: 625–646.
- [88] Mañosas F, Villadelprat J. A note on the critical period of polynomial systems[J]. *International Journal of Bifurcation & Chaos*, 2006, 3: 765–774.
- [89] Mañosas F, Villadelprat J. The bifurcation set of the period function of the dehomogenized Loud's center is bounded[J]. *Proceedings of the American Mathematical Society*, 2008, 136: 1631–1642.
- [90] Mañosas F, Villadelprat J. Criteria to bound the number of critical periods[J]. *Journal of Differential Equations*, 2009, 246: 2415–2433.
- [91] Mardesic P, Marín D, Villadelprat J. The period function of reversible quadratic centers[J]. *Journal of Differential Equations*, 2006, 224: 120–171.
- [92] Mardesic P, Rousseau R, Toni B. Linearization of isochronous centers[J]. *Journal of Differential Equations*, 1995, 121: 67–108.
- [93] Marín D, Villadelprat J. The criticality of reversible quadratic centers at the outer boundary of its period annulus[J]. *Journal of Differential Equations*, 2022, 332: 123–201.
- [94] Pleshkan I I. A new method of investigating the isochronicity of a system of two differential equations[J]. *Differential Equations*, 1969, 5: 796–802.
- [95] Rojas D, Villadelprat J. A critically result for polycycles in a family of quadratic reversible centers[J]. *Journal of Differential Equations*, 2018, 264: 6585–6602.
- [96] Romanovski V G, Chen X, Hu Z. Linearizability of linear systems perturbed by fifth degree homogeneous polynomials[J]. *Journal of Physics A Mathematical & Theoretical*, 2007, 40(22): 5905–5919.
- [97] Rothe F. The periods of the Volterra-Lotka system[J]. *Journal Für Die Reine Und Angewandte Mathematik*, 1985, 355: 129–138.
- [98] Rousseau C, Toni B. Local bifurcation of critical periods in vector fields with homogeneous nonlinearities of the third degree[J]. *Canadian mathematical bulletin*, 1993, 36(4): 473–484.
- [99] Rousseau C, Toni B. Local bifurcation of critical periods in the reduced Kukles system[J]. *Canadian Journal of Mathematics*, 1997, 49(2): 338–358.
- [100] Sabatini M. Characterizing isochronous centres by Lie brackets[J]. *Differential Equations Dynamical Systems*, 1997, 5: 91–99.
- [101] Sabatini M. A connection between isochronous Hamiltonian centres and the Jacobian Conjecture[J]. *Nonlinear Analysis Theory Methods & Applications*, 1998, 34: 829–838.
- [102] Sabatini M. On the period function of Liénard systems[J]. *Journal of Differential Equations*, 1999, 152: 467–487.
- [103] Schaaf R. A class of Hamiltonian systems with increasing periods[J]. *Journal Für Die Reine Und Angewandte Mathematik*, 1985, 363: 96–109.

- 
- [104] Schlomiuk D. Algebraic particular integrals, integrability and the problem of the center[J]. Transactions of the American Mathematical Society, 1993, 338(2): 799-841.
- [105] Sibirskii K S. On the number of limit cycles in the neighbourhood of a singular point[J]. Differential Equations, 1965, 1: 36-47.
- [106] Urabe M. Potential forces which yield periodic motions of a fixed period[J]. Indiana University Mathematics Journal, 1961, 10: 569-578.
- [107] Villarini M. Regularity properties of the period function near a center of a planar vector field[J]. Nonlinear Analysis: Theory, Methods & Applications, 1992, 19: 787-803.
- [108] Villadelprat J. On the reversible quadratic centers with monotone period function[J]. Proceedings of the American Mathematical Society, 2007, 135(8): 2555-2565.
- [109] Villadelprat J. The period function of the generalized Lotka-Volterra centers[J]. Journal of Mathematical Analysis and Applications, 2008, 341(2): 834-854.
- [110] Volokitin E P, Ivanov V V. Isochronicity and commutation of polynomial vector fields[J], Siberian Mathematical Journal, 1999, 40(1): 23-38.
- [111] Waldvogel J. The period in the Lotka-Volterra system is monotonic[J]. Journal of Mathematical Analysis and Applications, 1986, 114: 178-184.
- [112] Wang D. The critical points of the period function of  $x''-x^2(x-\alpha)(x-1)=0$  ( $0 \leq \alpha < 1$ )[J]. Nonlinear Analysis: Theory, Methods & Applications, 1987(11), 9: 1029-1050.
- [113] Wang Z, Chen X, Zhang W. Non-isochronicity of the center in polynomial Hamiltonian systems[J]. Nonlinear Analysis: Theory Methods & Applications, 2010, 73(1): 228-243.
- [114] Whittaker E T. A Treatise on the Analytical Dynamics of Particles and Rigid Bodies[M]. Cambridge: Cambridge University Press, 1993.
- [115] Yang L, Zeng X. The period function of potential systems of polynomials with real zeros[J]. Bulletin Des Sciences Mathématiques, 2009, 133(6): 555-577.
- [116] Zampieri G. On the periodic oscillations of  $\ddot{x} = g(x)$ [J]. Journal of Differential Equations, 1989, 78: 74-88.
- [117] Zhao Y. The monotonicity of period function for codimension four quadratic system  $Q_4$ [J]. Journal of Differential Equations, 2002, 185: 370-387.
- [118] Zhao Y. On the monotonicity of the period function of a quadratic system[J]. Discrete and continuous dynamical systems, 2005, 13(3): 795-810.
- [119] Zhao Y. The period function for quadratic integrable systems with cubic orbits[J]. Journal of Mathematical Analysis & Applications, 2005, 301( 2): 295-312.
- [120] Zolańdek H. Quadratic systems with centers and their perturbations[J]. Journal of Differential Equations, 1994, 109: 223-273.
- [121] Zou L, Chen X, Zhang W. Local bifurcations of critical periods for cubic Liénard equations with cubic damping[J]. Journal of Computational & Applied Mathematics, 2008, 222( 2): 404-410.